FINITE ELEMENT ANALYSIS OF ELASTIC-PLASTIC SOLIDS AT LARGE STRAIN

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ABSTRACT

A general purpose finite element program for the analysis of three dimensional elastic-plastic solids at large strains has been developed. In this work, this program is utilized to analyze a sample problem. The numerical results indicate the essential differences among the linear elastic, large strain elastic, and large strain elastic-plastic solutions. The issue of residual stresses is presented. The performance measure of computer software in general and finite element programs in particular is discussed.

1. INTRODUCTION

Among numerous theories of elastic-plastic solids, E.H. Lee's theory [1-6] is unique in the sense that the decomposition of total deformation into the elastic and the plastic parts is made at the deformation gradient level, i.e. $x_{i,K} = F^e_{ij}F^p_{jK}$, while the other theories start with the decomposition of strain (or strain rate) into the elastic and the plastic parts such as $E_{KL} = E^e_{KL} + E^p_{KL}$ in Green-Naghdi's theory [7]. Chiou *et al.* [8] made a comparison between these two theories and it was found that, in the case of simple tension test, the unloading curves are parallel on the plots of true (Cauchy) stress vs natural strain according to Lee's theory, and, on the other hand, according to Green-Naghdi's theory, the unloading curves are parallel on the plots of Piola-Kirchhoff stress vs Green-Lagrange strain but not on those of true stress vs natural strain. It can be shown that all the essential differences disappear, at least on a practical level, if the strains involved are infinitesimally small.

In the nonlinear analysis of solids, there are two kinds of nonlinearities - the material nonlinearity and the geometric nonlinearity. The material nonlinearity is basically due to the existence of a nonlinear relation between the stresses and the strains. The geometric nonlinearity implies that the strains involved are very large so that all the stress measures (Cauchy stress, Kirchhoff stress, first and second order Piola-Kirchhoff stresses, etc.) and the strain measures (engineering strain, natural strain, Green-Lagrange strain, etc.) are very much different in meaning and in numerical values.

There are many interesting problems and many unanswered questions in this field. It is felt that a general purpose finite element program with the capability of analyzing elastic-plastic solids at large strain might be a useful research tool to explore the field. With this belief, the finite element procedures for Lee's theory and Green-Naghdi's theory have been formulated [9]. A general purpose finite element program based on Lee's theory has been developed by Chiou *et al.* [10]. This program has been used to analyze elastic-plastic solids in simple tension test and simple shear test up to very large plastic strain. It was found that the agreements between the finite element solutions and the exact solutions are excellent. Also, Chiou has applied this program to analyze the tensile and compression tests of an aluminum ring and good agreement was found between the finite element solutions and the experimental data [11].

These lead to an interesting question: how can one measure the performance of computer software in general and finite element programs in particular? In this work, this finite element program is invoked to analyze a sample problem. The numerical results are presented and the issue of the performance measures of the computer software is discusssed.

2. CONSTITUTIVE RELATIONS

The general and detailed constitutive relations of E.H. Lee's elastic-plastic theory at finite strain have been derived by Lubarda and Lee [5]. In this work, let the special constitutive relations which are employed in the general purpose finite element program be listed as follows. First, the Helmholtz free energy density, Σ, as a function of the invariants of the elastic Cauchy-Green tensor, c_{ij}, may be expressed as

$$\Sigma = (\lambda+2\mu)(I_1-3)^2 / 8 - \mu(I_2-2I_1+3) / 2 \quad , \tag{1}$$

where λ and μ are the Lame's constants and

$$I_1 \equiv c_{kk} \quad , \quad I_2 \equiv (I_1^2 - c_{ij}c_{ij}) / 2 \quad . \tag{2}$$

Then the Kirchhoff stresses, τ_{ij}, can be written as

$$\tau_{ij} = (\lambda I_1-3\lambda-2\mu)c_{ij} / 2 + \mu c_{ik} c_{kj} \quad . \tag{3}$$

The Jaumann rate of the Kirchhoff stresses, τ_{ij}^{o}, can be obtained as

$$\tau_{ij}^{o} = A_{ijmn} D_{mn} \quad , \tag{4}$$

where

$$A_{ijmn} = (\lambda I_1-3\lambda-2\mu)\alpha_{ijmn} / 4 + \lambda c_{ij} c_{mn} + \mu\beta_{ijmn} / 2 + \mu(c_{im} c_{jn} + c_{in} c_{jm}) \quad , \tag{5}$$

$$\alpha_{ijmn} \equiv \delta_{im} c_{jn} + \delta_{in} c_{jm} + \delta_{jm} c_{in} + \delta_{jn} c_{im} \quad , \tag{6}$$

$$\beta_{ijmn} \equiv \delta_{im} c_{jk} c_{kn} + \delta_{in} c_{jk} c_{km} + \delta_{jm} c_{ik} c_{kn} + \delta_{jn} c_{ik} c_{km} \quad , \tag{7}$$

and **D** may be named the elastic component of the deformation rate tensor, whose definition and the associated discussions and derivations have been given by Lubarda and Lee [5]. Now eqn.(4) and its inverse may be written as

$$\boldsymbol{\tau}^{o} = \mathbf{A}\,\mathbf{D} \quad , \quad \mathbf{D} = \mathbf{A}^{-1} \boldsymbol{\tau}^{o} \quad . \tag{8}$$

Notice that **A** is a function of **c** and therefore, in view of eqn.(3), **A**, as well as $\mathbf{A}^{-1}$, is a function of the Kirchhoff stresses. The plastic deformation rate tensor, $\mathbf{d}^{p}$, is proposed to be

$$d_{ij}^{p} = \frac{9}{4HS^2}\,\tau'_{ij}\,\tau'_{mn}\,\tau^{\circ}_{mn} \quad , \qquad \textit{loading} \tag{9a}$$

$$= 0 \quad , \qquad \textit{unloading or neutral loading} \tag{9b}$$

where τ' is the Kirchhoff stress deviator, H is the strain hardening modulus, and S, the current yield strength, is equal to

$$S = \max\sqrt{1.5\tau_{ij}\tau_{ij}} \quad . \tag{10}$$

And "loading" is defined as the case of

$$1.5\tau'_{ij}\tau'_{ij} = S^2 \qquad \text{and} \qquad \tau'_{ij}\tau^{\circ}_{ij} > 0 \quad , \tag{11}$$

while "unloading" is

$$1.5\tau'_{ij}\tau'_{ij} < S^2 \qquad \textit{or} \qquad \tau'_{ij}\tau^{\circ}_{ij} < 0 \quad , \tag{12}$$

and "neutral loading" is

$$1.5\tau'_{ij}\tau'_{ij} = S^2 \qquad \text{and} \qquad \tau'_{ij}\tau^{\circ}_{ij} = 0 \quad . \tag{13}$$

Since the total deformation rate, $\mathbf{d}$, is the sum of $\mathbf{d}^p$ and $\mathbf{D}$, the constitutive relation in rate form can be expressed as

$$d_{ij} = (A^{-1}_{ijmn} + \frac{9I}{4HS^2}\tau'_{ij}\tau'_{mn})\tau^{\circ}_{mn} \quad , \tag{14}$$

where I is equal to 1 in loading and 0 in unloading or neutral loading. Let the inverse of eqn.(14) be written as

$$\tau^{\circ}_{ij} = a_{ijmn}\, d_{mn} \quad , \tag{15}$$

then the Truesdell stress rate tensor, $\overset{*}{\sigma}_{ij}$, can be obtained as [9,12]

$$\overset{*}{\sigma}_{ij} = \overset{*}{a}_{ijmn}\, d_{mn} \quad , \tag{16}$$

where

$$\overset{*}{a}_{ijmn} = a_{ijmn}/J - (\delta_{im}\sigma_{jn} + \delta_{in}\sigma_{jm} + \delta_{jm}\sigma_{in} + \delta_{jn}\sigma_{im})\,/\,2 \quad , \tag{17}$$

and J is the determinant of the deformation gradient.

3. FINITE ELEMENT PROCEDURES

The finite element procedures for the analysis of elastic-plastic solids at large strain have been given by Lee [9] and implemented by Chiou [11] and Chiou *et al.* [10]. In this work, only a few comments on the finite element procedures will be made. Equation (16), which links the Truesdell stress rate tensor and the deformation rate tensor, may be regarded as the stress-strain relation in rate form with $\mathbf{a}^*$ being the "slope" at a particular point in stress space. However, in nonlinear finite element analysis, one has to have a stress-strain relation in incremental form which enables the increments in displacements, strains, and stresses not to be infinitesimally small. Therefore, it is proposed to adopt the following incremental stress-strain relation

$$\Delta\overset{*}{\sigma}_{ij} = \overset{*}{\bar{a}}_{ijmn}\Delta\overset{*}{\varepsilon}_{mn} \quad , \tag{18}$$

where $\Delta\sigma^*$ and $\Delta\varepsilon^*$ are the incremental Truesdell stresses and the incremental Washizu strains, respectively [9,13]. It has been shown that

$$\boldsymbol{\sigma}^* = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{\sigma}^*}{\Delta t} \quad , \quad \mathbf{d} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{\varepsilon}^*}{\Delta t} \quad . \tag{19}$$

Therefore $\bar{\mathbf{a}}^*$ should be the average of $\mathbf{a}^*$ between two stress states (state 1 and state 2), i.e.,

$$\bar{\mathbf{a}}^* = average \quad \{ \mathbf{a}^*[\sigma(1)] \, , \mathbf{a}^*[\sigma(2)] \} \quad , \tag{20}$$

and the two stress states have to be weighted differently and carefully, especially when the stresses of state 1 (the beginning state) and state 2 (the ending state) are such that [9]

$$1.5\tau_{ij}'(1)\tau_{ij}'(1) < S^2 \quad , \quad 1.5\tau_{ij}'(2)\tau_{ij}'(2) > S^2 \quad . \tag{21}$$

It is seen that eqn.(20), in the limiting case, approaches eqn.(16). The calculation of stresses at state 2 involves approximation and iterations as discussed by Lee [9]. On the other hand, for those elements which never experience plastic deformation, one may directly calculate $\sigma(2)$ (after incremental displacements, $\Delta \mathbf{u}$, are obtained) as

$$x_{i,K} = \delta_{iK} + (u_i + \Delta u_i)_{,K} \quad , \tag{22}$$

$$\sigma_{ij} = \frac{2}{J} \frac{\partial \Sigma}{\partial C_{KL}} x_{i,K} \, x_{j,L} \quad . \tag{23}$$

This process does not involve the kind of approximation and iterations discussed above. Actually, it has been tried to solve the same elastic problems through two different routes as mentioned above and the results turn out to be the same -- the computer software has passed a perfomance measure test.

The nodal forces of a generic element, F_α, can be calculated as

$$F_\alpha(2) = \int \sigma_{ij}(2) B_{ij\alpha}(2) dv(2) \quad , \tag{24}$$

where $\mathbf{B}(2)$ is the matrix, at state 2, which links the displacement gradient and nodal displacements, i.e.,

$$u_{i,j} = B_{ij\alpha} \, U_\alpha \quad . \tag{25}$$

The sum over all the elements will yield all the components of the nodal forces at every nodal point , including the externally applied nodal forces and the reactive forces at the boundary due to displacement-specified boundary conditions. The calculation of nodal forces using eqn.(24) is an exact treatment in the sense that it has nothing to do with the approximations whatsoever involved in obtaining the incremental displacements, strains, and stresses. Therefore, the numerical results show that the total forces and the total moments due to the calculated nodal forces are all vanishing -- the computer software passes another test. However, the calculated nodal forces may turn out to be different from those expected and, when that happens, the differences will serve as the forcing terms in the next iteration until the differences are within the specified error tolerance.

4. SAMPLE PROBLEM

Consider a cantilever beam that originally occupies the space

$$V = \{ x,y,z \mid 0 \le x \le L, -b \le y \le b, 0 \le z \le h \} \quad . \tag{26}$$

The beam is fixed at x=0 and subjected to a downward loading at x=L. Because of the mirror symmetry with respect to the xz plane at y=0, one only has to analyze half of the beam. The finite element mesh for this cantilever beam consists of 80 8-node solid elements, 8 in the z-direction and 10 in the x-direction, and 198 nodal points as shown in Fig.1.

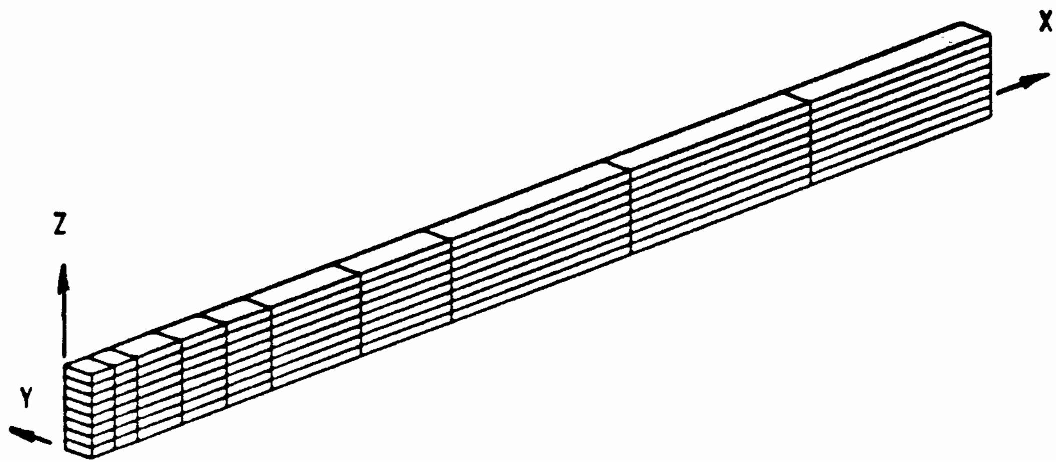


Figure 1. Undeformed Shape of the Cantilever Beam

The boundary conditions may be expressed as

(1) At $x=0$

$$u_x = u_y = u_z = 0 \quad , \tag{27}$$

(2) At $y=0$

$$u_y = 0 \ , \quad f_x = f_z = 0 \quad , \tag{28}$$

(3) At $x=L$

$$\Sigma f_z = -1*F \quad , \quad f_x = f_y = 0 \quad , \tag{29}$$

and the nodal forces at other boundary points are zero. The boundary conditions specified at $y=0$ are due to the mirror symmetry, the boundary conditions specified at the free end means a unit downward force is properly distributed to the nodal points at $x=L$, and F is the factor which goes up monotonically from zero to one in loading and back from one to zero in unloading. Upon the release of the applied loading, i.e., $F \to 0$, one may choose to keep the boundary conditions at the fixed end the same as those in eqn.(27), in other words, every point at $x=0$ is always stationary. This is referred to as Case 1 later.

On the other hand, in unloading,one may choose the boundary conditions at $x=L$ to be

$$u_x(0,y,z)=0, \quad u_y(0,0,z) = 0 \quad , \tag{30}$$

$$u_z(0,0,0)=0 \quad . \tag{31}$$

Note that $u_x=0$ in eqn.(30) is due to the mirror symmetry with respect to the yz plane at $x=0$, $u_y=0$ in eqn.(30) is also due to the mirror symmetry at $y=0$. However, $u_z = 0$ at the origin as specified in eqn.(31) is aimed at eliminating the rigid body motion and the computer results turn out to be that $f_z=0$ at the origin because it does not take a force to prevent the rigid body motion -- the computer program passes another performance measure test. The case with the boundary conditions specified in eqns.(30,31) for unloading is referred to as Case 2 later.

If elementary beam theory is employed, the bending stress, σ_{xx}, will be obtained readily as

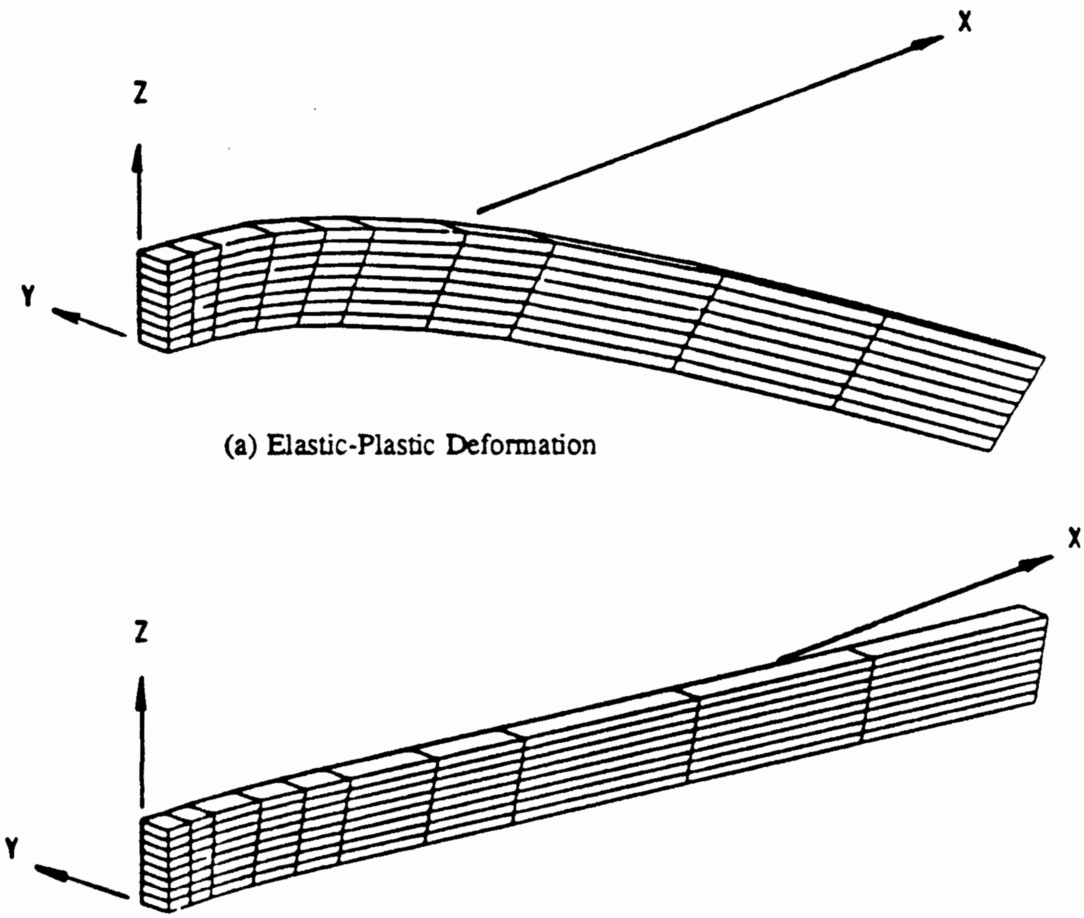
$$\sigma_{xx} = 12(L-x)(z-h/2)\ /\ bh^3 \quad . \tag{32}$$

The maximum bending stress occurs at $z=0$ and $z=h$ where

$$\sigma_m(x) = 6(L-x)\ /\ bh^2 \quad . \tag{33}$$

Later, for illustrative purposes, the stresses will be normalized with respect to $\sigma_m(x)$; also the normalized z-coordinate is defined to be

$$\bar{z} \equiv (2z-h)\ /\ h \quad . \tag{34}$$



(a) Elastic-Plastic Deformation

(b) Plastic (Residual) Deformation--Case 1

Figure 2. Deformed Shape of the Cantilever Beam

The material and geometric constants and other relevent parameters chosen in this work are

$$L = 10[L],\ \ b = 0.3[L],\ \ h = 1.0[L], \tag{35}$$

$$\lambda = 1000[F/L^2],\ \ \mu = 1000[F/L^2],\ \ H = 25[F/L^2], \tag{36}$$

$$S_0 = 100[F/L^2] \ , \tag{37}$$

where S_0 is the initial yield strength; the Young's modulus and the Poisson's ratio, corresponding to the Lame's constants, are

$$E = 2500[F/L^2] \ , \quad \nu = 0.25 \ . \tag{38}$$

The terminologies of Young's modulus, Poisson's ratio, and Lame's constants are still used here, however, it by no means implies that the work is in the scope of linear elasticity. On the other hand, when the strains involved are small, the solutions will reduce to the linear elastic ones.

5. NUMERICAL RESULTS

The graphic representations of the deformable body at the fully developed elastic-plastic state ($F=1$) and at the load released state (F is reduced to zero) are shown in Fig.2a and Fig.2b, respectively. The deformation was not exaggerated so that one may appreciate the large elastic-plastic strains by comparing Fig.1 and Fig.2a. Similarly, the differences between Fig.1 and Fig.2b are the plastic strains and the differences between Fig.2a and Fig.2b are the elastic strains. It is seen that the strains involved are indeed very large.

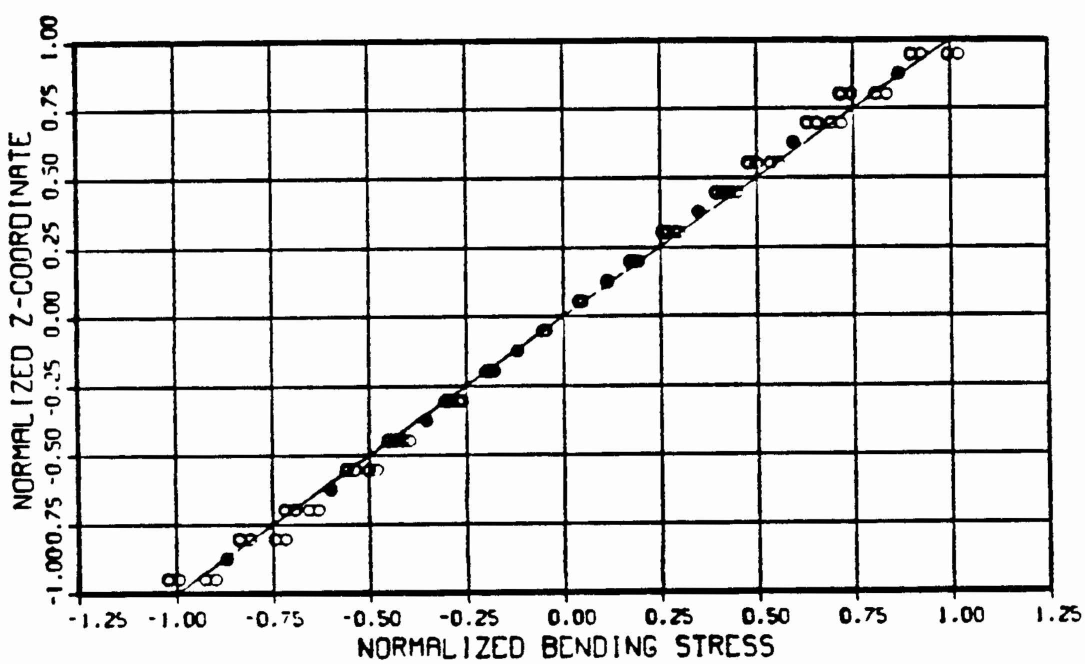


Figure 3. The Linear Elastic Solution
o -- Stresses at Gauss Points
• -- Average Stress at Centers

The normalized bending stresses at the Gauss points and centers of eight elements nearest to the fixed end are plotted as functions of the normalized z-coordinate in Fig.(3-7). The solid line represents the solution of the elementary beam theory, which serves as a reference for comparison. It is seen in Fig.3 that the linear elastic solution, which is obtained by multiplying the solution at F=0.01 by one hundred, matches the beam theory solution almost perfectly.

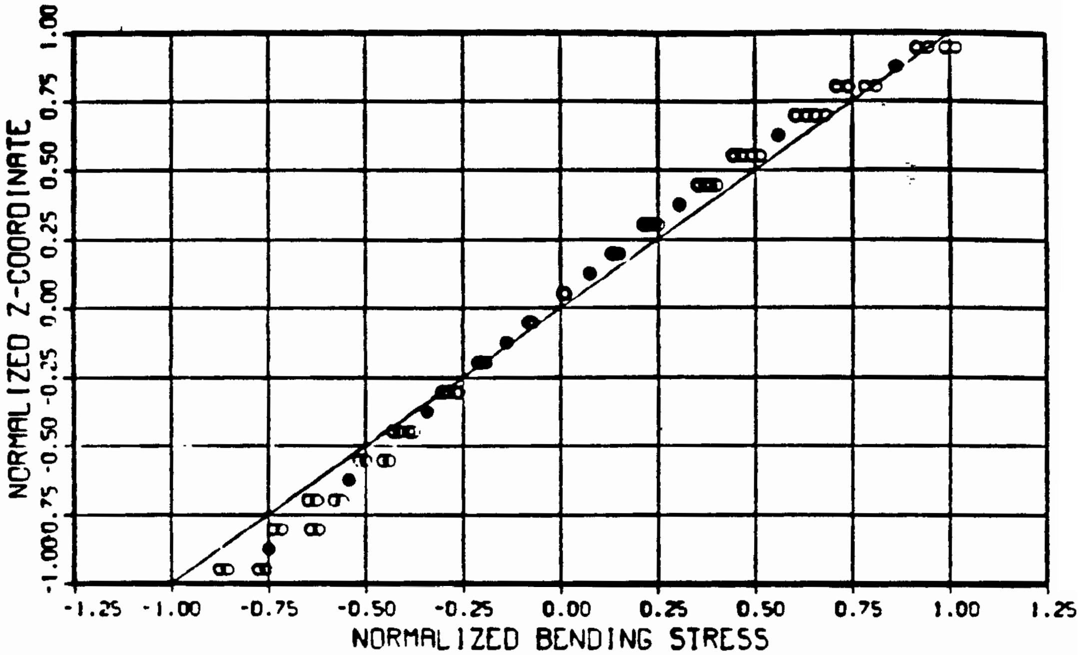


Figure 4. The Large Strain Elastic Solution
o -- Stresses at Gauss Points
● -- Average Stresses at Centers

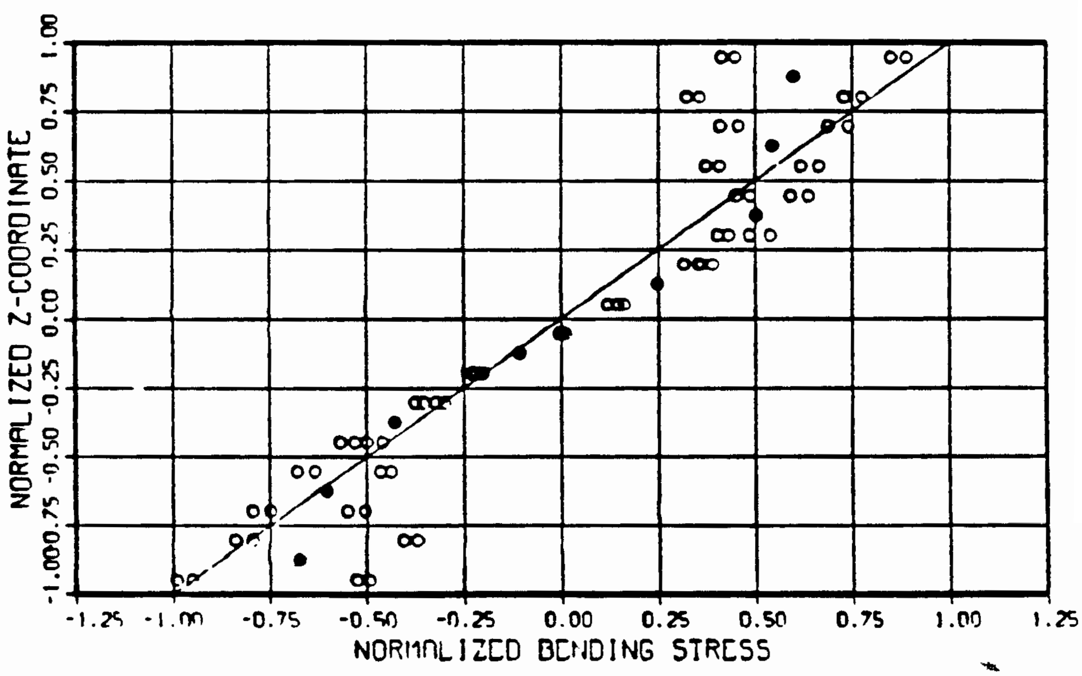


Figure 5. The large Strain Elastic-Plastic Solution
o -- Stresses at Gauss Points
● -- Average Stresses at Centers

The large strain elastic solution, presented in Fig.4, is obtained by setting the initial yield strength at a very large value so that the material will never experience plastic deformation. It is observed that (1) the stress distribution is no longer a straight line, and (2) the anti-symmetry between stresses in tension and in compression does not exist. Those observations are the effects of large strains.

The elastic-plastic solution, presented in Fig.5, indicates (1) the flatness of the (average) stress distribution near the top and the bottom of the beam, (2) the bending moment as a result of the bending stresses is smaller because the moment arms of the applied forces at the free end become smaller due to large deformation, and (3) the spread of stresses within an element becomes pronounced especially when the magnitude of the average stress becomes large.

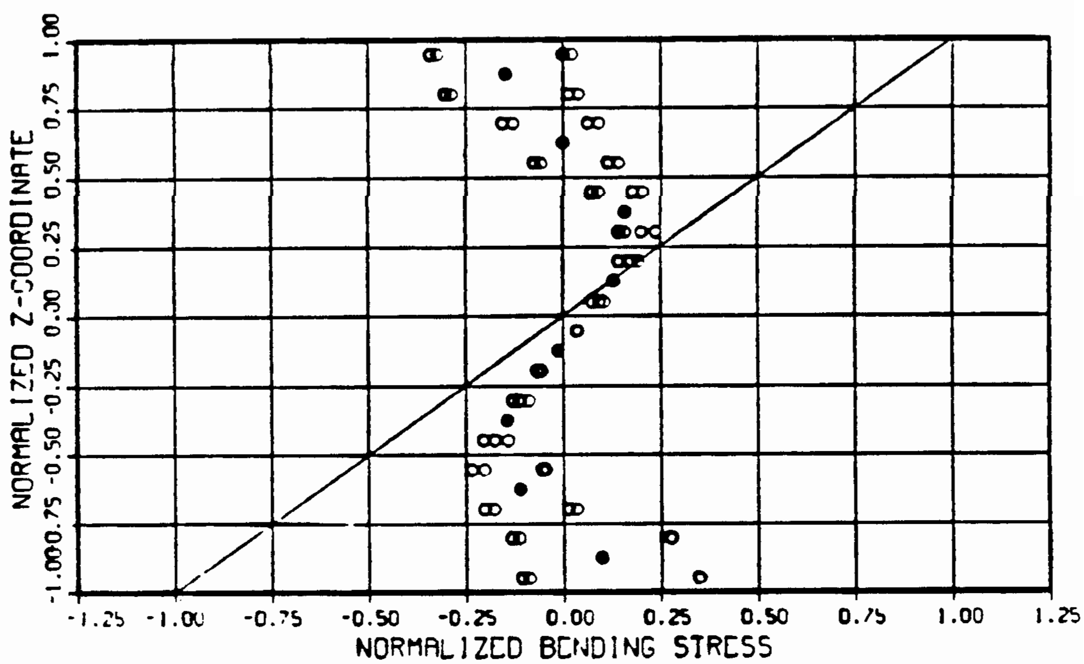


Figure 6. The Residual Stresses (Case 1)
o -- Stresses at Gauss Points
• -- Average Stresses at Centers

The residual stresses of Case 1 and Case 2 are shown in Fig.6 and Fig.7, respectively. The existence of those residual stresses in Case 1 is understood because the boundary conditions specified in eqn.(27) represent very severe constraints. However, it is a puzzle to observe that the residual stresses in Case 2, of which the boundary condition - eqn.(30) is only a representation of mirror symmetries, are as pronounced as those in Case 1.

The magnitude of the stress at the Gauss point, which is nearest to the origin, is shown as a function of the deflection of the beam at the free end, $[-u_z(L,0,0)]$, in Fig.8 for the entire loading and unloading processes. The differences between Case 1 and Case 2 can be observed. The contrast between the stiffness in the elastic range and that in the plastic range is indicated also.

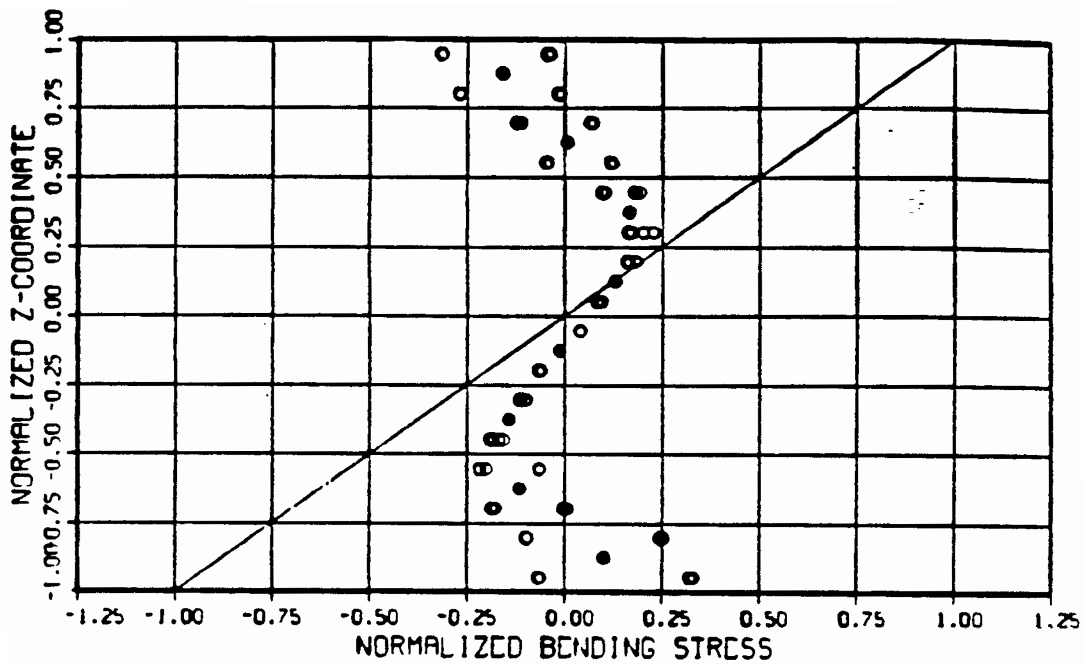


Figure 7. The Residual Stresses (Case 2)
o -- Stresses at Gauss Points
● -- Average Stresses at Centers

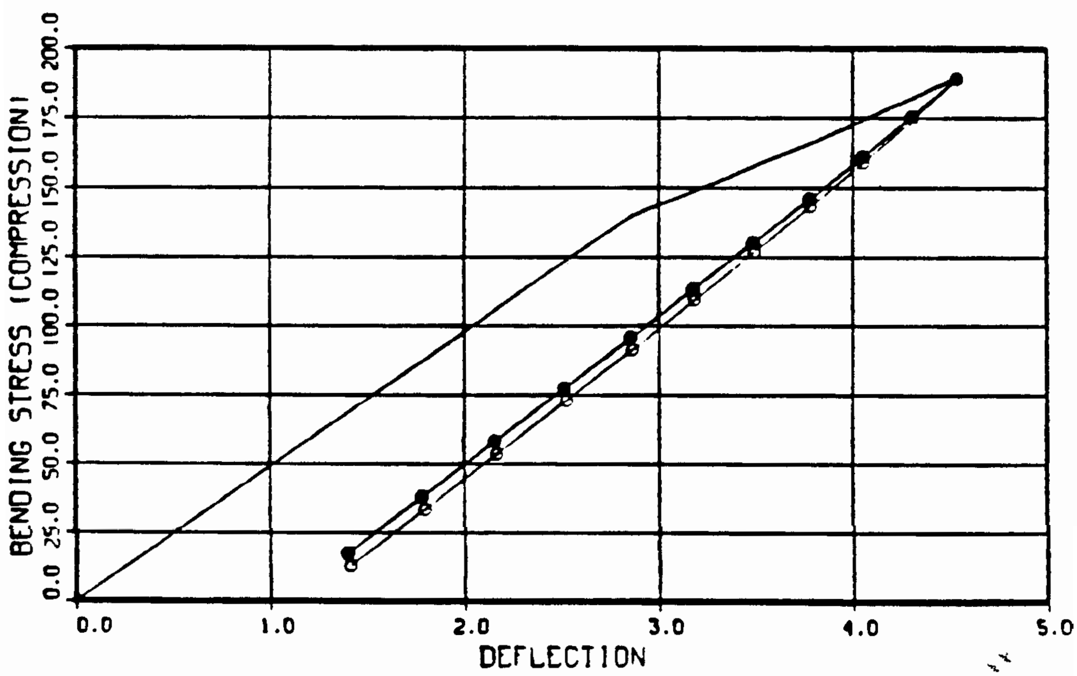


Figure 8. The Bending Stress -- Deflection Curve in Loading and Unloading Processes
● -- Case 1　　　o -- Case 2

6. DISCUSSIONS

The performance measure of computer software becomes an important issue as computer users are concerned about the quality of software which they use. Recently, a set of techniques, collectively called software engineering, has evolved to deal with computer software as an engineering product that requires planning, analysis, design, implementation, testing, and maintenance [14]. In this work, it is only attempted to discuss this broad issue from a particular view point : for a given computer software, what and how does one measure its performance? A list of items which need to be measured together with some discussions and/or comments is given as follows (with no intention to claim that this list is complete) :

(1) Existence of syntex errors, bugs, or defects

(2) Efficiency

Efficiency is usually measured by the cpu time, number of lines of code, storage requirement, etc. For softwares which are run on a vector processing computer, efficiency also means how well the softwares take the advantage of vector processsing. The finite element program used in this work is run on Cyber 205, a supercomputer, and efforts have been made to vectorize the code. It is worthwhile to say that how to develop a finite element code which fully utilizes the capability of a vector processing computer is a very interesting and important topic.

(3) User friendliness

For certain software, this is perhaps the most important issue. However, the measure of user friendliness, to some extent, is a subjective judgement.

(4) Effectiveness / Correctness

For finite element programs, this is the measure of validity. It is felt that, at least, one should compare the finite element solutions with the exact analytical solutions of some simple problems. That is why this finite element program has been used to analyze simple tension and simple shear problems up to very large elastic-plastic strains[10]. In general, if there is no exact solution to compare with, it may be proper to check with common sense or even intuition. Also, cross examinations may be helpful --- in this work, a few cross examinations have been made and discussed in Section 3 and Section 4. The measure of effectiveness/correctness will be more accurate if more different kinds of cross examinations are carried out.

(5) Modularity / Capability for improvement

This is the measure of potentiality for the further development of the software. Usually, this concerns the developer, instead of the user, of the software.

(6) Generality / Domain of applications

For finite element programs, this is the measure of numbers of different types of loadings, boundary conditions, elements, material properties, etc. in the programs.

(7) Sophistication / Theoretical Profoundness

This is a measure of the level of profoundness associated with the software. In finite element analysis, the level of profoundness increases from elasticity to plasticity, from one dimensional through two dimensional to three dimensional, from small strain through large deflection to large strain, from static analysis to dynamic analysis, etc. It is worthwhile to say that each increase in the level of profoundness brings a better understanding and, sometimes, more unanswered questions in the field.

(8) Artificial intelligence

To define and measure the level of artificial intelligence is perhaps the most challenging task in the performance measure of computer software. Usually, the finite element programs existing nowadays are regarded as having no, or at most limited, intelligence, simply because heuristics is hardly involved.

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